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On Fractional Schrödinger Equation and Its Application

Avik Kumar Mahata

Abstract— In the note, recent efforts to derive fractional quantum mechanics are recalled. Time-space fractional Schrödinger equation with and without a Non Local term also been discussed. Applications of a fractional approach to the Schrödinger equation for free electron and potential well are discussed as well.

Index Terms— Caputo Fractional Derivative, Fractional Calculus, Fractional Schrödinger Equation, Space Time Fractional Schrödinger Equation, Time-Space fractional Schrödinger like equation, Free Electron, Potential Well.

1. INTRODUCTION

The fractality concept in Quantum Mechanics developed throughout the last decade. The term fractional Quantum Mechanics was first introduced by Laskin [1]. The path integrals over the Lévy paths are defined and fractional quantum and statistical mechanics have been developed via new fractional path integrals approach. A fractional generalization of the Schrödinger equation, new relation between the energy and the momentum of non-relativistic fractional quantum-mechanical particle has been established [1]. The concept of fractional calculus, originated from Leibniz, has gained increasing interest during last two decades (see e.g. [2] and activity of the Fracalmo research group [3]). But the concept of fractional calculus was originally introduced by Leibniz, which has gained special interest in recent time. Later Laskin [4] studied the properties of the fractional Schrödinger equation.

Hermiticity of the fractional Hamilton operator was proved and also established the parity conservation law for fractional quantum mechanics. As physical applications of the fractional Schrödinger equation he found the energy spectra of a hydrogenlike atom (fractional "Bohr atom") and of a fractional oscillator in the semiclassical approximation. An equation for the fractional probability current density is developed and discussed and also discussed the relationships between the fractional and standard Schrödinger equations. Later E.Ahmed et al. studied on the order of fractional Schrödinger equation [5]. Muslih & Agrawal [6] studied the Schrödinger equation is studied in a fractional space. Specifically, they present the model of a hydrogen-like fractional atom called "fractional Bohr atom." Recently Herrmann has applied the fractional mechanical approach to several particular problems [7-12]. Some other cases of the fractional Schrödinger equation were discussed by Naber [13] and Ben Adda & Cresson [14].

• *Avik Kumar Mahata is currently pursuing masters degree program in Material Science and Engineering in National Institute of Technology, Trichy, Tamilnadu, INDIA, PH-+919786177547, E-mail: avik894@gmail.com*

2. FRACTIONAL CALCULUS

Fractional Calculus is a field of mathematic study that grows out of the traditional definitions of the calculus integral and derivative operators in much the same way fractional exponents is an outgrowth of exponents with integer value. Since the appearance of the idea of differentiation and integration of arbitrary (not necessary integer) order there was not any acceptable geometric and physical interpretation of these operations for more than 300 year. In [15], it is shown that geometric interpretation of fractional integration is"'Shadows on the walls"' and its Physical interpretation is" 'Shadows of the past"'. I will discuss different definition of fractional differentiation and integration below,

2.1 Different Definition

i) L. Euler (1730)

Euler generalized the formula

$$\frac{d^n x^m}{dx^n} = m(m-1)\ldots\ldots(m-n+1)x^{(m-n)}$$

By using the following property of Gamma function,

$$\Gamma(m+1) = m(m-1)\ldots\ldots(m-n+1)\,\Gamma(m-n+1)$$

To obtain;

$$\frac{d^n x^m}{dx^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)}x^{(m-n)}$$

Gamma function is defined as follows,

$$\Gamma(z) = \int_0^\infty e^{-t}t^{(z-1)}dt\,, \quad \mathrm{Re}(z)>0$$

ii) J.B.J Fourier (1820-22)

By means of integral representation,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(z)dz\int_{-\infty}^{\infty}\cos(\mathrm{px}-\mathrm{pz})\mathrm{dp}$$

He wrote,

$$\frac{d^n f(x)}{dx^n} = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(z)dz\int_{-\infty}^{\infty}\cos\left(\mathrm{px}-\mathrm{pz}+\frac{\mathrm{n}\pi}{2}\right)\mathrm{dp}$$

iii) J.Liovilli (1832-1855)

The third definition of Liovilli includes *Fractional Derivative*,

$$\frac{d^\mu F(x)}{dx^\mu} = \frac{(-1)^\mu}{h^\mu}\left(F(x)\frac{\mu}{1}F(x+h)+\frac{\mu(\mu-1)}{1.2}F(x+2h)-\cdots\right)$$

iv) G.F.B Riemann

His definition of fractional integral is,

$$D^{-\nu}f(x) = \frac{1}{\Gamma(\nu)}\int_c^x (x-t)^{(\nu-1)}f(t)dt + \psi(t)$$

v) Riemann-Liovilli definition

The popular definition of fractional calculus is this which shows joining of the above two definitions,

$$D_\alpha^t f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n\int_a^t \frac{f(\tau)d\tau}{(t-\tau)^{\alpha-n+1}}$$

vi) The Caputo fractional derivative

The Caputo definition of the fractional derivative follows the inverted sequence of operations. An ordinary differentiation is followed by the fractional integration,

$${}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(\alpha-n)}\int_a^t \frac{f^{(n)}(\tau)d\tau}{(t-\tau)^{(\alpha-n+1)}},$$

$(n-1 \le \alpha < n)$.

vii) **The fractional Schrödinger equation**.

The most general fractional Schrödinger equation may be obtained when,

$$\frac{\partial}{\partial t}\to\frac{\partial^\alpha}{\partial t}\ and\ \frac{\partial}{\partial x}\to\frac{\partial^\beta}{\partial x}$$

However, special care need to be undertaken in order to introduce canonically conjugate observables $\hat{X}$ and $\hat{P}$.

3. Space Time Fractional Schrödinger Equation

The Feynman path integral formulation of quantum mechanics is based on a path integral over Brownian paths. In diffusion theory, this can also be done to generate the standard diffusion equation; however, there are examples of many phenomena that are only properly described when non-Brownian paths are considered. When this is done, the resulting diffusion equation has factional derivatives [16,17]. Due to the strong similarity between the Schrödinger equation and the standard diffusion equation one might expect modifications to the Schrödinger equation generated by considering non-Brownian paths in the path integral derivation. This gives the time-factional, space-fractional and space–time-factional Schrödinger equation [4, 19, and 20].

In Quantum Mechanics the famous Schrödinger is given by (in one dimension),

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + V(x,t)\psi(x,t)$$

where ψ(x, t) and V (x, t) denote the wave function and the potential function, respectively. Feynman and Hibbs [21] reformulated the Schrödinger equation by use of a path integral approach considering the Gaussian probability distribution.

Later the time space fractional Schrödinger equation [22] obtained by Dong and Xu
(in one dimension);

$$(i\hbar)^\beta D_{0,t}^\beta\psi(x,t) = \frac{\hbar^\beta \boldsymbol{D}_\alpha}{E_p T_p^\beta}(-\hbar^2\Delta)^{\alpha/2}\psi(x,t) + \frac{\hbar^\beta V(x,t)}{E_p T_p^\beta}\psi(x,t) \quad \textbf{(1)}$$

where $\psi(x,t)$ and $V(x,t)$ are wave function and potential energy respectively, $D_\alpha = \frac{\mathbf{1}}{\mathbf{2}\boldsymbol{m}_\alpha}$ is "fractional quantum diffusion coefficient" with physical dimension,

$$D_\alpha = erg^{1-\alpha} \times cm^\alpha \times sec^\alpha$$

$$D_\alpha = \frac{1}{2m} \text{ for } \alpha = 2$$

And ,

$$L_p = \sqrt{\frac{G\hbar}{c^3}},\ T_p = \sqrt{\frac{G\hbar}{c^5}}, M_p = \sqrt{\frac{c\hbar}{G}},\ E_p = M_p c^2 \quad \textbf{(2)}$$

are Plank length, time, mass and energy. And G and c are Gravitational constant and speed of light respectively and $D_{0,t}^\beta$ Caputo Fractional Derivative of order **β** and $(-\hbar^2\Delta)^{\alpha/2}$ is quantum Riesz fractional derivative [see 23,24] defined by the following,

$$(-\hbar^2\Delta)^{\alpha/2}\psi(x,t) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}(dp)e^{\frac{ipx}{\hbar}}|p|^\alpha \times \int_{-\infty}^{\infty} e^{\frac{ipx}{\hbar}}\psi(x,t)dx,\ (1<\alpha\leq 2) \quad \textbf{(3)}$$

4. Time-Space fractional Schrödinger like equation

Lenzi and Oliveira et al. [25] analyze the following integer Schrödinger equation with a nonlocal term,

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = \frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + \int_0^t d\bar{t}\int_{-\infty}^{\infty} d\bar{x}U(x-\bar{x},t-\bar{t})\psi(\bar{x},t) \quad \textbf{(4)}$$

where the last term represents a nonlocal term acting on the system and the kernel *U(x, t).* According to literatures [22] and [25], when a nonlocal term act on the fractional quantum system equation (1) can be written as the following,

$$(i\hbar)^\beta D_{0,t}^\beta\psi(x,t) = \frac{\hbar^\beta \boldsymbol{D}_\alpha}{E_p T_p^\beta}(-\hbar^2\Delta)^{\alpha/2}\psi(x,t) + \frac{\hbar^\beta V(x,t)}{E_p T_p^\beta}\psi(x,t) + \frac{\hbar^\beta}{E_p T_p^\beta}\int_0^t d\bar{t}\int_{-\infty}^{\infty} d\bar{x}U(x-\bar{x},t-t)\psi(\overline{x,t}) \quad \textbf{(5)}$$

5. Solution of simple system by Time Fractional Schrodinger Equation

5.1 Free Electron

The time fractional Schrödinger equation for a free particle is given by,

$$(iT_p)^\beta D_t^\beta = -\frac{L_p^2}{2N_m}\partial_x^2 \quad \textbf{(6)}$$

To solve this equation, apply a Fourier transform on the spatial coordinate,

$$F(\psi(x,t)) = \psi(\lambda,t)$$

$$(iT_p)^\beta D_t^\beta\psi = \frac{(L_p\lambda)^2}{2N_m}\psi \quad \textbf{(7)}$$

The resulting equation can be rearranged and the results of the second appendix can be used. Namely, the identity for the Mittag-Leffler (26,27) function with a complex argument,

$$D_t^\beta \psi = \frac{(L_p\lambda)^2}{2N_m(iT_p)^\beta}\psi \quad \textbf{(8)}$$

And we get,

$$\psi = \frac{\psi_0}{\beta}\{e^{-i\omega^{\frac{1}{\beta}}t} - \beta F_\beta\big(\omega(-i)^\beta, t\big)\} \quad \textbf{(9)}$$

where $\omega = \frac{(L_p\lambda)^2}{2N_m {T_p}^\beta}$ In Eq.(9) the first term is oscillatory, and the second is decay in the time variable. Inverse Fourier transforming gives the final solution,

$$\psi(x,t) = F^{-1}\psi(\lambda,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x}\frac{\psi_0}{\beta}\{e^{-i\omega^{\frac{1}{\beta}}t} - \beta F_\beta\big(\omega(-i)^\beta, t\big)\}d\lambda \quad \textbf{(10)}$$

This can be broken into two parts, a Schrödinger like piece divided by **β**, and a decay term **β** that goes to zero as time goes to infinity. Note that as goes to one the decay term goes to zero and the Schrödinger like term becomes the non-fractional Schrödinger term. ψ_0 may be chosen so that the initial probability is one. Due to the decay term in the solution one may ask what happens to the total probability as time goes to infinity. We can manipulate a few more steps and obtain the final probability like the following,

$$\lim_{t\to\infty}\int_{-\infty}^{\infty}\psi(x,t)\bar{\psi}(x,t)dx = \frac{1}{\beta^2} \quad \textbf{(11)}$$

Since **β** less than one, the total probability increases over time to the limiting value of Eqn. (11).

5.2 Potential Well

The energy levels for the potential well can be computed. Since the Hamiltonian is time dependent the energy levels will also be time dependent.

$$E_n(t) = \int_0^a \psi^* i\hbar\partial_t\psi dx \quad (12)$$

Or,

$$E_n(t) = \frac{i\hbar}{\beta^2}\{e^{-i\omega^{\frac{1}{\beta}}t} - \beta F_\beta\big(\omega(-i)^\beta, t\big)\}\partial_t\{e^{-i\omega^{\frac{1}{\beta}}t} - \beta F_\beta\big(\omega(-i)^\beta, t\big)\} \quad \textbf{(13)}$$

The interesting result is when time goes to infinity.

$$E_n(\infty) = \hbar\boldsymbol{\lambda}_{\boldsymbol{n}}^{\boldsymbol{1/\beta}}/\beta^2 T_p \quad \textbf{(14)}$$

This is the same energy spectrum that is obtained for the non-fractional Schrödinger equation except for the factor of $1/\boldsymbol{\beta^2}$ and the exponent of 1/**β** . Since **β** is less than one the spacing between energy levels is greater than that given by the non-fractional Schrödinger equation. In fact the smaller the value of **β** the greater the difference between energy levels. And at t = 0 the spacing between the energy levels is that same as that of the non-fractional Schrödinger equation.

6. Conclusion

Complex systems (CS) are frequent in nature. Mathematical methods can be helpful in understanding them. Motivated by the properties of CS we concluded that Fractional order systems are more suitable for describing CS for the following reasons: First they are more natural in describing fractal systems. Second they are more natural in describing systems with memory and delay. Third they are more natural in describing non-local systems. This may have an impact on the entanglement problem in quantum mechanics. Motivated by the theory of fractal space time, fractional order generalization for Schrodinger equations are presented. Fractional order automatically include nonlocality which is a known property of quantum systems. It is interesting that the bright idea of Nottale, Ord and El-Naschie [28,29] namely fractal space time offers a unifying scheme for such diverse fields as quantum mechanics, earthquakes and fractional order equations.

Also it is interesting to see the wide range of applications of nonlinear systems e.g. algebra [30], game theory [31], economics [32] an addition to physics [33].

In conclusion, we can emphasize that though the basis of fractional quantum mechanics is already formulated, the range of solved problems is very narrow. Therefore, it can be considered as an interesting field for further exploration.

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